A note on semilattices of groups

By SÁNDOR LAJOS in Budapest

Let S be a semigroup. Following the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [2] we shall say that S is a semilattice of groups if it is the set-theoretical union of a set of mutually disjoint subgroups G_α $(\alpha \in \Lambda)$:

(1) $$S = \bigcup_{\alpha \in \Lambda} G_\alpha$$

such that, for any couple α, β in Λ, the products $G_\alpha G_\beta$ and $G_\beta G_\alpha$ are both contained in the same G_γ $(\gamma \in \Lambda)$.

Recently the author proved several ideal-theoretic characterizations of semigroups that are semilattices of groups (see [3]—[10]). In this note we shall prove two new criteria for a semigroup S to be a semilattice of groups.

Theorem 1. *A semigroup S is a semilattice of groups if and only if the set of all bi-ideals of S is a semilattice under the multiplication of subsets.*

Proof. First suppose that S is a semigroup being a semilattice of groups. Then, by a recent result of the author [5]

(2) $$B_1 \cap B_2 = B_1 B_2$$

for any couple of bi-ideals of S. This implies that every bi-ideal B of S is globally idempotent and the condition

(3) $$B_1 B_2 = B_2 B_1$$

holds for any two bi-ideals B_1, B_2 of S. Hence the set of all bi-ideals of S becomes a multiplicative semilattice.

Conversely, let S be a semigroup where bi-ideals form a commutative idempotent semigroup. Then $L^2 = L$ and $R^2 = R$ for any left ideal L, and any right ideal R of S, respectively. Furthermore, the condition

(4) $$LR = RL$$

holds for every left ideal L and every right ideal R of S. Then, by a result of J. CALAIS

[1], S is a regular semigroup. Finally, a recent criterion of the author [6] guarantees that S is a semilattice of groups. Our Theorem 1 is completely proved.

The proof of the following result is quite similar to that of Theorem 1.

Theorem 2. *A semigroup S is a semilattice of groups if and only if all the quasi-ideals of S form a semilattice under the multiplication of subsets.*

Theorem 1, Theorem 2, and the author's earlier results imply the following statement.

Theorem 3. *For a semigroup S the following conditions are pairwise equivalent:*

(A) *S is a semilattice of groups.*

(B) $L\cap R = LR$ *for every left ideal L and every right ideal R of S.*

(C) $B\cap L = LB$ for *every bi-ideal B and left ideal L of S.*

(D) $B\cap R = BR$ *for every bi-ideal B and right ideal R of S.*

(E) $L\cap Q = LQ$ *for every left ideal L and quasi-ideal Q of S.*

(F) $Q\cap R = QR$ *for every quasi-ideal Q and right ideal R of S.*

(G) $B\cap Q = BQ$ *for every bi-ideal B and quasi-ideal Q of S.*

(H) $B\cap Q = QB$ *for every bi-ideal B and quasi-ideal Q of S.*

(I) $Q_1\cap Q_2 = Q_1Q_2$ *for every two quasi-ideals of S.*

(J) $B_1\cap B_2 = B_1B_2$ *for every couple of bi-ideals of S.*

(K) $L_1\cap L_2 = L_1L_2$ *and* $R_1\cap R_2 = R_1R_2$ *for every two left ideals L_1, L_2 and for every two right ideals R_1, R_2 of S.*

(L) $I\cap L = LI$ *and* $I\cap R = IR$ *for every left ideal L, right ideal R, and two-sided ideal I of S.*

(M) $L\cap R = LSR$ *for every left ideal L and right ideal R of S.*

(N) $Q_1\cap Q_2 = Q_1SQ_2$ *for every two quasi-ideals of S.*

(O) $B_1\cap B_2 = B_1SB_2$ *for every couple of bi-ideals of S.*

(P) *The intersection of every k quasi-ideals of S is equal to their product (k is an arbitrary fixed positive integer >1).*

(Q) *The intersection of every k bi-ideals of S is equal to their product (k is an arbitrary fixed positive integer >1).*

(R) *The set of all quasi-ideals of S is a multiplicative semilattice.*

(S) *The set of all bi-ideals of S is a commutative band under the multiplication of subsets.*

(T) *S is centric and every principal left ideal of S is globally idempotent.*

(U) *The intersection of every k left ideals of S is equal to their product and the same holds for right ideals too* ($k>1$).

(V) *S is a completely regular*[1]) *inverse semigroup.*

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[1]) A semigroup S is said to be *completely regular* if to every element a in S there exists x of S such that $axa=a$ and $ax=xa$.

(W) *S is a regular duo*[2]) *semigroup.*

(X) $A \cap B = AB$ *for every two* (m, n)*-ideals of* S *(m, n are arbitrary positive integers).*

(Y) *The intersection of every* k (m, n)*-ideals of* S *is equal to their product* $(k, m, n$ *are arbitrary fixed positive integers,* $k > 1)$.

(Z) $A \cap B = AB$ *for every* $(0, n)$*-ideal* A *of* S *and every* $(m, 0)$*-ideal* B *of* S *(m, n are arbitrary fixed positive integers).*

For the definition and fundamental properties of (m, n)-ideals of semigroups, see the author's papers [11] and [12].

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[2]) A semigroup S is called a *duo semigroup* if every one-sided (left or right) ideal of S is two-sided.